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CYCLES IN WEIGHTED GRAPHS

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A weighted graph is one in which each edge e is assigned a nonnegative number $w(e)$, called the weight of e. The weight $w(G)$ of a weighted graph G is the sum of the weights of its edges. In this paper, we prove, as conjectured in [2], that every 2-edge-connected weighted graph on n vertices contains a cycle of weight at least $2w(G)/(n-1)$. Furthermore, we completely characterize the 2-edge-connected weighted graphs on n vertices that contain no cycle of weight more than $2w(G)/(n-1)$. This generalizes, to weighted graphs, a classical result of Erdős and Gallai [4].

1. Introduction

Let $G=(V,E)$ be a simple graph (without loops or multiple edges). G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e; in case we want to distinguish the underlying graph, we write $w_G(e)$ instead of $w(e)$. For any subgraph H of G, $V(H)$ and $E(H)$ denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H)=\sum_{e\in E(H)} w(e).$$

A cycle (path) is called an *optimal cycle* (*path*) if it is a cycle (path) of maximum weight. For a vertex v, the *weighted degree* $w(v)$ of v is the sum of the weights of the edges incident with v. The *neighbour set* of v is denoted by $N(v)$ and the *degree* of v by $d(v)$. An unweighted graph can be regarded as a weighted graph in which each edge e is assigned weight $w(e)=1$. Thus, in an unweighted graph, $w(v)$ is simply the degree of v. A (u,v)-*path* is a path connecting the two vertices u and v. A set S of vertices of G is called a *vertex cut* of G if the removal of S leaves a graph with more components than G; S is an *s-vertex cut* if $|S|=s$. A graph is *separable* if it has a 1-vertex cut; otherwise it is *nonseparable*. It follows from the definitions that every 2-connected graph is nonseparable, and that every nonseparable graph on at least three vertices is 2-connected.

Throughout the paper, we exclude the trivial case in which each edge is of zero weight. The following two conjectures were proposed in [2].

Conjecture 1.1. *Let G be a weighted graph on n vertices. Then G contains a path of weight at least $2w(G)/n$.*

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Conjecture 1.2. *Let G be a 2-edge-connected weighted graph on n vertices. Then G contains a cycle of weight at least $2w(G)/(n-1)$.*

It was shown in [2] that Conjecture 1.2 implies Conjecture 1.1. Recently, Frieze, McDiarmid, and Reed [5] proved that Conjecture 1.1 is true. In this paper, we prove Conjecture 1.2, thereby generalizing, to weighted graphs, a classical result of Erdős and Gallai [4]. Thus, we prove

Theorem 1.3. *Let G be a 2-edge-connected weighted graph on n vertices. Then G contains a cycle of weight at least $2w(G)/(n-1)$.*

Furthermore, we completely characterize the 2-edge-connected weighted graphs on n vertices that contain no cycle of weight more than $2w(G)/(n-1)$. Our characterization is based on the notation of a tritree.

Definition 1.4. A spanning tree T of a graph G is called a *tritree* if every fundamental cycle of T in G is a triangle.

Definition 1.5. Let G be a weighted graph and $\mathbf{H}$ a set of subgraphs of G (not necessarily distinct). If there is an assignment of a positive real number α_H to each $H \in \mathbf{H}$ such that, for every $e \in E(G)$,

$$w(e) = \sum\{\alpha_H : e \in H \in \mathbf{H}\},$$

then we say that G is a *weighted union* of the members of $\mathbf{H}$, and write

$$G = \sum_{H \in \mathbf{H}} \alpha_H H.$$

Definition 1.6. A 2-edge-connected weighted graph G on n vertices is *cycle-extremal* if its optimal cycles are of weight precisely $2w(G)/(n-1)$.

We shall prove

Theorem 1.7. *A 2-edge-connected weighted graph is cycle-extremal if and only if it is a weighted union of tritrees.*

Theorems 1.3 and 1.7 are valid for weighted complete graphs ([2], Theorem 6). We quote the result here, since it will be needed in the sequel.

Theorem 1.8. *Let G be a weighted complete graph on n vertices, where $n \geq 3$. Then the maximum weight of a cycle of G is at least $2w(G)/(n-1)$, with equality if and only if $G = \sum_{T \in \mathbf{T}} \alpha_T T$, where $\mathbf{T}$ is the set of spanning stars of G.*

The following result was also proved in [2]; it will be used later.

Theorem 1.9. *Let G be a 2-connected weighted graph and d a real number. Let x and y be two distinct vertices of G. If $w(v) \geq d$ for all $v \in V(G) \setminus \{x, y\}$, then G contains an (x, y)-path of weight at least d.*

To conclude this section, we give one more definition and a lemma.

Definition 1.10. Let G be a weighted graph and $e \in E(G)$. Define G_e to be the weighted graph obtained from G by contracting the edge e and, for each pair of multiple edges in the resulting graph, deleting the edge of smaller weight (or either, if they have equal weight).

Lemma 1.11. *Let G be a 2-connected weighted graph and $x \in V(G)$. If $|V(G)| > 3$, then there is $y \in N(x)$ such that G_{xy} is 2-edge-connected.*

Proof. Let $y \in N(x)$. If G_{xy} is not 2-edge-connected, then, since G is 2-connected, there must be $z \in N(x) \setminus \{y\}$ such that $d(z) = 2$. Then G_{xz} is 2-edge-connected. ∎

2. Proof of Theorem 1.3

Lemma 2.1. *Let G be a 2-connected weighted graph and P an optimal path in G, with ends x and y. Then there is a cycle C in G such that*

$$w(C) > w(P) \quad \text{or} \quad w(C) \geq w(x) + w(y).$$

Proof. Let

$$P = v_0 v_1 \ldots v_l,$$

where $v_0 = x$ and $v_l = y$, and define

$$S = \{v_i \,:\, xv_i \in E\} \quad \text{and} \quad T = \{v_i \,:\, v_{i-1}y \in E\}.$$

Note that, by optimality of P,

$$(2.1) \qquad w(v_{i-1}v_i) \geq w(xv_i),\ v_i \in S \quad \text{and} \quad w(v_{i-1}v_i) \geq w(v_{i-1}y),\ v_i \in T.$$

Case 1: $S \cap T \neq \emptyset$.
If $y \in S \cap T$ and $w(xy) > 0$, the cycle $C = Pyx$ has weight

$$w(C) = w(P) + w(xy) > w(P).$$

Otherwise, for $v_i \in S \cap T$, define

$$C_i = v_0 v_1 \ldots v_{i-1} v_l v_{l-1} \ldots v_1 v_0.$$

Then the cycles C_i together cover the edges $v_{i-1}v_i$, $v_i \notin S \cap T$, $|S \cap T|$ times, the edges $v_{i-1}v_i$, $v_i \in S \cap T$, $|S \cap T| - 1$ times, and the edges xv_i and $v_{i-1}y$, $v_i \in S \cap T$, once. Therefore,

$$\sum\{w(C_i) \,:\, v_i \in S \cap T\} =$$
$$(|S \cap T| - 1)w(P) + \sum\{w(v_{i-1}v_i) \,:\, v_i \notin S \cap T\} + \sum\{w(xv_i) + w(v_{i-1}y) \,:\, v_i \in S \cap T\}.$$

Using (2.1), it follows that

$$\begin{aligned}
&\sum\{w(C_i) : v_i \in S \cap T\} - (|S \cap T| - 1)w(P) \geq \\
&\sum\{w(v_{i-1}v_i) : v_i \in S \setminus T\} + \sum\{w(v_{i-1}v_i) : v_i \in T \setminus S\} \\
&\qquad + \sum\{w(xv_i) : v_i \in S \cap T\} + \sum\{w(v_{i-1}y) : v_i \in S \cap T\} \\
&\geq \sum\{w(xv_i) : v_i \in S\} + \sum\{w(v_{i-1}y) : v_i \in T\} \\
&= w(x) + w(y).
\end{aligned}$$

Thus, if C is the cycle C_i of maximum weight, C has the required property.

Case 2: $S \cap T = \emptyset$.

Following Bondy and Locke [3; p.112], we define a *vine* on P to be a set $\mathbf{Q} = \{Q_j : 1 \leq j \leq m\}$ of internally-disjoint paths in G such that

(1) $P \cap Q_j = \{a_j, b_j\}$, $1 \leq j \leq m$;

(2) $x = a_1 < a_2 < b_1 \leq a_3 < b_2 \leq a_4 < \ldots \leq a_m < b_{m-1} < b_m = y$.

where a_j and b_j are the ends of Q_j and $<$ denotes precedence on P. Since G is 2-connected, it follows easily that there is a vine $\mathbf{Q}$ on P. We choose $\mathbf{Q}$ so that

(i) m is as smalls as possible;

(ii) subject to (i), $\left|\bigcup_{j=1}^{m-1} V_j\right|$ is as small as possible, where V_j denotes the set of internal vertices of the segment $P[a_{j+1}, b_j]$ of P, $1 \leq j \leq m-1$.

We claim that the cycle

$$C = P \cup \left(\bigcup_{j=1}^{m} Q_j\right) - \left(\bigcup_{j=1}^{m-1} V_j\right)$$

has the required property. Denote the vertex following a_m on P by a_m^+. By the choice of $\mathbf{Q}$, the assumption that $S \cap T = \emptyset$, and (2.1),

$$\begin{aligned}
w(C) &= \sum\{w(v_{i-1}v_i) : v_i \notin V_j \cup \{b_j\},\ 1 \leq j \leq m-1\} + \sum\{w(Q_j) : 1 \leq j \leq m\} \\
&\geq \sum\{w(v_{i-1}v_i) : v_i \in S \setminus \{b_1\}\} + \sum\{w(v_{i-1}v_i) : v_i \in T \setminus \{a_m^+\}\} + w(xb_1) + w(a_m y) \\
&\geq \sum\{w(xv_i) : v_i \in S \setminus \{b_1\}\} + \sum\{w(v_{i-1}y) : v_i \in T \setminus \{a_m^+\}\} + w(xb_1) + w(a_m y) \\
&\geq \sum\{w(xv_i) : v_i \in S\} + \sum\{w(v_{i-1}y) : v_i \in T\} \\
&= w(x) + w(y).
\end{aligned}$$

This completes the proof of the lemma. ∎

Proof of Theorem 1.3. We apply induction on n, and then on $\binom{n}{2} - |E(G)|$. If $n = 3$ or $|E(G)| = \binom{n}{2}$, the result follows from Theorem 1.8. Suppose now that $n > 3$, $|E(G)| > \binom{n}{2}$ and the result holds for all 2-edge-connected graphs G' with $|V(G')| < n$ or with $|V(G')| = n$ and $|E(G')| > |E(G)|$.

If G is separable, let $G = G_1 \cup G_2$, where $|V(G_1) \cap V(G_2)| = 1$. Set $n_i = |V(G_i)|$, $i = 1, 2$. By the induction hypothesis, G_i contains a cycle of weight at least $2w(Gi)/(n_i - 1)$, $i = 1, 2$. Therefore, if C is an optimal cycle of G,

$$w(C) \geq w(C_i) \geq \frac{2w(G_i)}{n_i - 1}, \qquad i = 1,\ 2.$$

Hence,

$$2w(G) = 2w(G_1) + 2w(G_2) \leq (n_1 - 1)w(C) + (n_2 - 1)w(C) = (n-1)w(C),$$

which gives $w(C) \geq 2w(G)/(n-1)$. If there is $x \in V(G)$ such that $w(x) \leq w(G)(n-1)$, then, by Lemma 1.11, there is $y \in N(x)$ such that G_{xy} is 2-edge-connected. Note that

$$w(G_{xy}) \geq w(G) - w(x) \geq \frac{n-2}{n-1} w(G).$$

By the induction hypothesis, G_{xy} contains a cycle C' of weight

$$w(C') \geq \frac{2w(G_{xy})}{n-2} \geq \frac{2w(G)}{n-1}.$$

However, either C' is a cycle in G or it can be extended to a cycle in G. Therefore, we may assume that G is 2-connected and that

$$w(v) > \frac{w(G)}{n-1} \quad \text{for every} \quad v \in V(G). \tag{2.2}$$

Since G is not complete, let $xy \notin E(G)$. Add xy to G with zero weight and let G' denote the resulting weighted graph. By the induction hypothesis, G' contains a cycle C' of weight at least $2w(G')/(n-1) = 2w(G)/(n-1)$. If $xy \notin C'$, then C' is also a cycle of G, which completes the proof. Suppose that $xy \in C$. Since $w(xy) = 0$, the path $C - xy$ in G is of weight at least $2w(G)/(n-1)$. Let P be an optimal path in G, so $w(P) \geq 2w(G)/(n-1)$. It follows from Lemma 2.1, using (2.2), that G contains a cycle of weight more than $2w(G)/(n-1)$. This completes the proof of Theorem 1.3. ∎

3. Results on Tritrees

In this section, we give some results on tritrees. They will be needed in the proof of Theorem 1.7. The first two propositions were proved in [1].

Proposition 3.1. *Let T be a tritree of a graph G. Then any cycle of G contains at most two edges of T.*

Proposition 3.2. *Let G be a graph with no 2-vertex cut. Then each tritree of G is a spanning start.*

Proposition 3.3. *Let G be a weighted graph and G' a weighted graph obtained by adding to G edges of weight zero. If $G' = \sum_{T \in \mathbf{T}} \alpha_T T$ for some set $\mathbf{T}$ of tritrees of G', then $\mathbf{T}$ is also a set of tritrees of G and $G = \sum_{T \in \mathbf{T}} \alpha_T T$.*

Proof. Since α_T is positive and the new edges in G' have zero weight, none of the new edges belongs to any tritree in $\mathbf{T}$. Hence, each tritree in $\mathbf{T}$ is also a tritree of G, and $G = \sum_{T \in \mathbf{T}} \alpha_T T$. ∎

Proposition 3.4. *Let G be a weighted graph on n vertices. If $G = \sum_{T \in \mathbf{T}} \alpha_T T$ for some set $\mathbf{T}$ of tritrees of G, then*

$$\sum_{T \in \mathbf{T}} \alpha_T = \frac{w(G)}{n-1}.$$

Proof. By the definition,

$$w(G) = \sum_{T \in \mathbf{T}} \alpha_T |E(T)| = \sum_{T \in \mathbf{T}} \alpha_T (n-1) = (n-1) \sum_{T \in \mathbf{T}} \alpha_T,$$

which gives the required equality. ∎

Proposition 3.5. *Let G be a weighted graph on n vertices. If $G = \sum_{T \in \mathbf{T}} \alpha_T T$ for some set $\mathbf{T}$ of tritrees of G, then, for any $e \in E(G)$,*

$$w(e) \leq \frac{w(G)}{n-1},$$

with equality if and only if every tritree in $\mathbf{T}$ contains e.

Proof. Let

$$\mathbf{T}' = \{T \in \mathbf{T} \ : \ T \text{ contains } e\}.$$

Then, using Proposition 3.4, we have

$$w(e) = \sum_{T \in \mathbf{T}'} \alpha_T \leq \sum_{T \in \mathbf{T}} \alpha_T = \frac{w(G)}{n-1},$$

with equality if and only if $\mathbf{T}' = \mathbf{T}$ as required. ∎

Proposition 3.6. *Let G be a weighted graph on n vertices and C an optimal cycle in G. If $G = \sum_{T \in \mathbf{T}} \alpha_T T$ for some set $\mathbf{T}$ of tritrees of G, then $w(C) \leq 2w(G)/(n-1)$, with equality if and only if $|E(C) \cap E(T)| = 2$ for every $T \in \mathbf{T}$.*

Proof. By Proposition 3.1, C contains at most two edges of T, for any $T \in \mathbf{T}$. Thus, using Proposition 3.4,

$$w(C) \leq \sum_{T \in \mathbf{T}} 2\alpha_T = 2 \sum_{T \in \mathbf{T}} \alpha_T = \frac{2w(G)}{n-1},$$

with equality if and only if $|E(C) \cap E(T)| = 2$ for every $T \in \mathbf{T}$, as required. ∎

4. Cycle-Extremal Graphs

Proposition 4.1. *If G is a cycle-extremal graph, then $w(v) \geq w(G)/(n-1)$ for all $v \in V(G)$.*

Proof. Let $v \in V(G)$. If $w(v) < w(G)/(n-1)$, by Lemma 2.1, there is $y \in N(v)$ such that G_{vy} is 2-edge-connected, where

$$w(G_{vy}) \geq w(G) - w(y) > \frac{(n-2)w(G)}{n-1}.$$

By Theorem 1.3, G_{vy} has a cycle C' of weight

$$w(C') \geq \frac{2w(G_{vy})}{n-2} > \frac{2w(G)}{n-1}.$$

But, either C' is a cycle in G or it can be extended to a cycle in G. This contradicts the fact that G is cycle-extremal. Therefore, $w(v) \geq w(G)/(n-1)$ for all $v \in V(G)$. ∎

Definition 4.2. Let G_1 and G_2 be two weighted graphs such that $|V(G_1) \cap V(G_2)| = 1$. Set $n_i = |V(G_i)|$, $i = 1, 2$. If $w(G_1)/(n_1 - 1) = w(G_2)/(n_2 - 1)$, then $G_1 \cup G_2$ is called the *1-sum* of G_1 and G_2.

Proposition 4.3. *Let G be a separable cycle-extremal graph. Then G is a 1-sum of two cycle-extremal graphs.*

Proof. Since G is separable, let $G = G_1 \cup G_2$, where $|V(G_1) \cap V(G_2)| = 1$. Set $n_i = |V(G_i)|$, $i = 1, 2$. We prove that $w(G_1)/(n_1 - 1) = w(G_2)/(n_2 - 1)$ and that G_i is cycle-extremal, $i = 1, 2$. Let C be an optimal cycle in G and C_i an optimal cycle in G_i, $i = 1, 2$. Since G is cycle-extremal,

$$2w(G) = (n-1)w(C) = (n_1-1)w(C)+(n_2-1)w(C) \geq (n_1-1)w(C_1)+(n_2-1)w(C_2).$$

By Theorem 1.3, $(n_i - 1)w(C_i) \geq 2w(G_i)$, $i = 1, 2$, and so

$$2w(G) \geq 2w(G_1) + 2w(G_2) = 2w(G).$$

Thus, the above inequalities are all equalities, and

$$\frac{2w(G_1)}{n_1 - 1} = w(C_1) = w(C) = w(C_2) = \frac{2w(G_2)}{n_2 - 1}.$$

This completes the proof of Proposition 4.3. ∎

Proposition 4.4. *Let G be a 2-connected cycle-extremal graph on n vertices. Then*
(i) each edge of G lies in an optimal cycle, and
(ii) any two nonadjacent vertices are connected by a path of weight at least $2w(G)/(n-1)$.

Proof. (i) Let $e \in E(G)$. Replace $w(e)$ by $w(e) + \epsilon$. By Theorem 1.3, the resulting weighted graph G_ϵ has a cycle C_ϵ of weight

$$w(C_\epsilon) \geq \frac{2w(G_\epsilon)}{n-1} = \frac{2w(G)}{n-1} + \frac{2\epsilon}{n-1}.$$

Since G has no cycle of weight more than $2w(G)/(n-1)$, C_ϵ must pass through e. Letting $\epsilon \to 0$, and noting that the number of cycles through e is finite, we deduce that some optimal cycle of G must pass through e.

(ii) Let $uv \notin E(G)$. Join u and v by an edge e of weight ϵ, and denote the resulting graph by G_ϵ. The above argument now shows that u and v are connected in G by a path of weight at least $2w(G)/(n-1)$. ∎

Proposition 4.5. *Let G be a 2-connected cycle-extremal graph of positive weight with a 2-vertex cut $\{x, y\}$. Then $xy \in E(G)$ and $w(xy) = w(G)/(n-1)$.*

Proof. Let $G = H_1 \cup H_2$, where $V(H_1) \cap V(H_2) = \{x, y\}$ and $E(H_1) \cup E(H_2) = E(G) \setminus \{xy\}$. So

$$w(H_1) + w(H_2) = w(G) - w(xy), \tag{4.1}$$

where $w(xy) = 0$ if $xy \notin E(G)$. Let $n_i = |V(H_i)|$ and P_i an (x, y)-path of maximum weight in H_i, $i = 1, 2$. Since $H_i + xy$ is 2-connected and, by Proposition 4.1, $w(v) \geq w(G)/(n-1)$ for all $v \in V(H_i) \setminus \{x, y\}$, $i = 1, 2$, it follows from Theorem 1.9 that

$$w(P_i) \geq \frac{w(G)}{n-1}, \quad i = 1,\ 2. \tag{4.2}$$

Put $G_i = H_i + xy$ with $w_G(xy) = w(P_{3-i})$, $i = 1, 2$. So

$$w(G_i) = w(H_i) + w(P_{3-i}), \quad i = 1,\ 2.$$

Let C be an optimal cycle in G and C_i an optimal cycle in G_i, $i = 1, 2$. By Theorem 1.3,

$$(n_i - 1)w(C_i) \geq 2w(G_i) = 2w(H_i) + 2w(P_{3-i}), \quad i = 1,\ 2. \tag{4.3}$$

Since either C_i is a cycle in G or it can be converted to a cycle in G by replacing xy with P_{3-i}, we have

$$w(C) \geq w(C_i), \quad i = 1,\ 2. \tag{4.4}$$

Thus,

$$nw(C) = (n_1 - 1)w(C) + (n_2 - 1)w(C) \geq (n_1 - 1)w(C_1) + (n_2 - 1)w(C_2).$$

By (4.3) and (4.1),

$$nw(C) \geq 2(w(H_1) + w(H_2) + w(P_1) + w(P_2)) = 2(w(G) - w(xy) + w(P_1) + w(P_2)).$$

Hence,

$$w(xy) \geq w(G) + w(P_1) + w(P_2) - \frac{nw(C)}{2}.$$

Using (4.2), and noting that $w(C) = 2w(G)/(n-1)$, we have

$$w(xy) \geq \frac{w(G)}{n-1}. \tag{4.5}$$

This implies, since $w(G) > 0$, that $xy \in E(G)$. Now, $P_1 \cup \{x, y\}$ is a cycle in G, which gives that

$$w(P_1) + w(xy) \leq w(C) = \frac{2w(G)}{n-1}.$$

This together with (4.2) and (4.5) implies that $w(xy) = w(G)/(n-1)$, and completes the proof of Proposition 4.5. ∎

Definition 4.6. Let G_1 and G_2 be two nonseparable weighted graphs such that $V(G_1) \cap V(G_2) = \{x, y\}$ and $E(G_1) \cap E(G_2) = \{xy\}$. If

$$\frac{w(G_1)}{n_1 - 1} = w(xy) = \frac{w(G_2)}{n_2 - 1},$$

where $n_i = |V(G_i)|$, $i = 1, 2$, then $G_1 \cup G_2$ is called the *2-sum* of G_1 and G_2.

Proposition 4.7. *Let G be a 2-connected cycle-extremal graph of positive weight with a 2-vertex cut $\{x, y\}$. Then G is a 2-sum of two cycle-extremal graphs.*

Proof. Define G_i, n_i, C_i, $i = 1$, 2, as in the proof of Proposition 4.3. Since equalities hold in (4.3) and (4.4),

$$\frac{2w(G_1)}{n_1 - 1} = w(C_1) = w(C) = w(C_2) = \frac{2w(G_2)}{n_2 - 1}.$$

This implies that G_i is a cycle-extremal graph, $i = 1$, 2. Moreover, from Proposition 4.5, $xy \in E(G)$ and $w(xy) = w(G)/(n-1) = w(C)/2$. Hence

$$\frac{w(G_1)}{n_1 - 1} = w(xy) = \frac{w(G_2)}{n_2 - 1}.$$

Therefore, G is the 2-sum of the cycle-extremal graphs G_1 and G_2. ∎

5. Proof of Theorem 1.7

By Proposition 3.6 and Theorem 1.3, it suffices to prove

Theorem 5.1. *If G is a cycle-extremal graph, then $G = \sum_{T \in \mathbf{T}} \alpha_T T$ for some set $\mathbf{T}$ of tritrees of G.*

Proof. By induction on n, and then on $\binom{n}{2} - |E(G)|$. For $n = 3$ or $|E(G)| = \binom{n}{2}$, the result follows from Theorem 1.8. Suppose now that $n > 3$ and $|E(G)| < \binom{n}{2}$. If $w(G) = 0$, the result is trivially true by taking $\mathbf{T} = \emptyset$. Thus, we may assume that $w(G) > 0$.

If G is separable, then, by Proposition 4.3, G is a 1-sum of cycle-extremal graphs, G_1 and G_2. By the induction hypothesis,

$$G_1 = \sum_{i=1}^{m_1} \alpha'_j T'_j \quad \text{and} \quad G_2 = \sum_{k=1}^{m_2} \alpha''_k T''_k.$$

By Proposition 3.4 and the definition of 1-sum,

$$G_1 = \sum_{i=1}^{m_1} \alpha'_j = \sum_{k=1}^{m_2} \alpha''_k.$$

Denote this common value by α, and let

$$\alpha_{jk} = \frac{\alpha'_j \alpha''_k}{\alpha} \quad \text{and} \quad T_{jk} = T'_j \cup T''_k, \quad 1 \le j \le m_1 \text{ and } 1 \le k \le m_2.$$

It is clear that T_{jk} is a tritree of G. We claim that

$$G = \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \alpha_{jk} T_{jk}. \tag{5.1}$$

Let $e \in E(G)$. We may suppose, without loss of generality, that $e \in E(G_1)$. Then

$$\begin{aligned} w(e) &= \sum\{\alpha'_j : e \in E(T'_j),\ 1 \le j \le m_1\} \\ &= \frac{1}{\alpha}\left(\sum_{k=1}^{m_1} \alpha''_k\right) \sum\{\alpha'_j : e \in E(T'_j),\ 1 \le j \le m_1\} \\ &= \sum\left\{\frac{\alpha'_j\alpha''_k}{\alpha} : e \in E(T'_j \cup T''_k),\ 1 \le j \le m_1,\ 1 \le k \le m_2\right\} \\ &= \sum\{\alpha_{jk} : e \in E(T_{jk}),\ 1 \le j \le m_1,\ 1 \le k \le m_2\}. \end{aligned}$$

Therefore, (5.1) holds, as claimed.

If G is 2-connected and has a 2-vertex cut $\{x, y\}$, then, by Proposition 4.7, G is a 2-sum of cycle-extremal graphs G_1 and G_2. By the induction hypothesis,

$$G_1 = \sum_{i=1}^{m_1} \alpha'_j T'_j \text{ and } G_2 = \sum_{k=1}^{m_2} \alpha''_k T''_k.$$

By proposition 3.4 and the definition of 2-sum,

$$\sum_{j=1}^{m_1} \alpha'_j = w(xy) = \sum_{k=1}^{m_1} \alpha''_k, \tag{5.2}$$

xy being the common edge of G_1 and G_2. By Proposition 3.5 and 4.5, each of the tritrees T'_j and T''_k, $1 \le j \le m_1$ and $1 \le k \le m_2$, includes the edge xy. Furthermore, since $\{x, y\}$ is a 2-vertex cut of G, $T'_j \cup T''_k$ is a tritree of G, $1 \le j \le m_1$ and $1 \le k \le m_2$. As before, denote the common value in (5.2) by α, and let

$$\alpha_{jk} = \frac{\alpha'_j\alpha''_k}{\alpha} \text{ and } T_{jk} = T'_j \cup T''_k, \quad 1 \le j \le m_1,\ 1 \le k \le m_2.$$

Then

$$G = \sum_{j=1}^{m_1}\sum_{k=1}^{m_2} \alpha_{jk} T_{jk},$$

as required by the theorem. Therefore, we may assume, noting that $n > 3$, that

$$G \text{ is 3-connected.} \tag{5.3}$$

Since G is not complete, and by Proposition 4.4, the optimal paths of G are of weight at least $2w(G)/(n-1)$. If $w(v) > w(G)/(n-1)$ for all $v \in V(G)$, then, by Lemma 2.1, G contains a cycle of weight more than $2w(G)/(n-1)$. This is impossible. Therefore, we may assume that there is a vertex $u \in V(G)$ such that $w(u) \le w(G)/(n-1)$. This implies, by Proposition 4.1, that

$$w(u) = \frac{w(G)}{n-1}. \tag{5.4}$$

Let $G' = G - u$. Then

$$w(G') = w(G) - w(u) = \frac{n-2}{n-1} w(G).$$

Now let C' be an optimal cycle in G'. Since $C' \subseteq G$,

$$w(C') \le \frac{2w(G)}{n-1}.$$

On the other hand, by (5.3), G' is 2-connected. It follows from Theorem 1.3 that

$$w(C') \ge \frac{2w(G')}{n-2} \ge \frac{2w(G)}{n-1}.$$

Consequently,

(5.5) G' is a 2-connected cycle-extremal graph.

Let $x,\, y \in N(u)$. If $xy \notin E(G)$, then, by (5.5) and Proposition 4.4(ii), x and y are connected by a path P' of weight

$$w(P') \ge \frac{2w(G')}{n-2} = \frac{2w(G)}{n-1}.$$

However, $xP'yux$ is a cycle in G, and G has no cycle of weight more than $2w(G)/(n-1)$. Therefore, $w(ux) = w(uy) = 0$. This shows that, for all $x,\, y \in N(u)$,

(5.6) $$xy \notin E(G) \Longrightarrow w(ux) = w(uy) = 0.$$

Let $x,\, y \in N(u)$. If $xy \in E(G)$, then, by (5.5) and Proposition 4.4(i), there is an optimal cycle C' in G' passing through xy. Since G' is cycle-extremal,

$$w(C') = \frac{2w(G')}{n-2} = \frac{2w(G)}{n-1}.$$

Let C be the cycle obtained from C' by replacing the edge xy with the path xuy. Then $C \subseteq G$, and hence $w(ux) + w(uy) \le w(xy)$, for otherwise C is of weight more than $2w(G)/(n-1)$. Therefore, for all $x,\, y \in N(u)$ and $xy \in E(G)$,

(5.7) $$w(ux) + w(uy) \le w(xy).$$

Set

$$X = \{v \in N(u) : w(uv) > 0\}.$$

By (5.6), X induces a complete graph K in G. If $|X| = 2$, Let C be the triangle $xyux$; if $|X| > 2$, Let C be a Hamilton cycle of K. In either case, by (5.7), $w(C) \ge 2w(u)$. It follows from (5.4) that $w(C) \ge 2w(G)/(n-1)$. But, G contains no cycle of weight more than $2w(G)/(n-1)$. So all inequalities in (5.7) are equalities. Furthermore, if we set $w(xy) = 0$ for $x,\, y \in N(u)$ and $xy \notin E(G)$, then we also have, by (5.6), that $w(ux) + w(uy) = 0 = w(xy)$. Therefore,

(5.8) $$w(ux) + w(uy) = w(xy) \text{ for all } x,\, y \in N(u).$$

Our next goal is to prove that

(5.9) $$G' = \sum_{T' \in \mathbf{T}'} \alpha_{T'} T',$$

where each $T' \in \mathbf{T}'$ is a spanning star of G'.

If G' has no 2-vertex cut, then (5.9) follows from the induction hypothesis and Proposition 3.2. Suppose, now, that G' has a 2-vertex cut. Let $\{x, y\}$ be a 2-vertex cut of G'. By (5.5) and Proposition 4.5, $xy \in E(G)$. Let $G' = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{x, y\}$ and $E(G_1) \cap E(G_2) = \{xy\}$. Choose the 2-vertex cut $\{x, y\}$ so that $|V(G_1)|$ is as small as possible. Then

(5.10) any 2-vertex cut of G' is contained in $V(G_2)$.

To see this, let $\{z_1, z_2\}$ be a 2-vertex cut of G'. If $\{z_1, z_2\} \notin V(G_2)$, suppose, without loss of generality, that $z_1 \in V(G_1) \setminus \{x, y\}$. By the choice of $\{x, y\}$, it must be that $z_2 \in V(G_2) \setminus \{x, y\}$. But, $G_1 - z_1$ is a connected graph containing xy and $G_2 - z_2$ is a connected graph containing xy. Therefore, $G - \{z_1, z_2\}$ is connected. This contradiction establishes (5.10).

Since G is 3-connected, there is $z \in V(G_1) \setminus \{x, y\}$ and $zu \in E(G)$. Let H be the graph obtained from G' by adding edges of weight zero to join z with all vertices in $N(u)$ which are not adjacent to z in G. It is clear that any cycle in H is either a cycle in G or can be converted to a cycle in G by replacing at most two edges of weight zero with two edges incident with u. This implies that no cycle in H is of weight more than

$$\frac{2w(G)}{n-1} = \frac{2w(G')}{n-2} = \frac{2w(H)}{n-2}.$$

It follows from Theorem 1.3 that H is cycle-extremal. By the induction hypothesis,

$$H = \sum_{T \in \mathbf{T}_H} \alpha_T T,$$

where $\mathbf{T}_H$ is a set of tritrees of H. Moreover, by (5.3) and the structure of H, H has no 2-vertex cuts because any 2-vertex cut would contain z and so contradict (5.10). Therefore, by Proposition 3.2, each tritree in $\mathbf{T}_H$ is a spanning star, and (5.9) follows from Proposition 3.3.

Recall $X = \{v \in N(u) : w(uv) > 0\}$. If $|X| \le 2$, then, by (5.8) and (5.4), there is an edge $e \in E(G')$ incident with a vertex in X with $w(e) = w(u) = w(G)/(n-1)$. By Proposition 3.5, each tritree T' in (5.9) contains the edge e, and hence T' is a star centered at a vertex in X. If $|X| > 2$, let C be a Hamilton cycle of the graph induced by X. Then, as mentioned before, $w(C) = 2w(u) = 2w(G)/(n-1)$. Consequently, by Proposition 3.6, $|E(C) \cap E(T')| = 2$ for every tritree T' in (5.9). This also yields that T' is centered at a vertex in X. Therefore, setting $V(G') = \{v_1, v_2, \ldots, v_{n-1}\}$ and $X = \{v_1, v_2, \ldots, v_m\}$, we may rewrite (5.9) as

$$G' = \sum_{i=1}^{m} \alpha_i T_i',$$

where T_i' is the star tritree centered at vertex v_i, $1 \le i \le m$. Put

$$\alpha_i = 0, \quad m+1 \le i \le n-1.$$

Then, by Definition 1.5, for any i and j, $1 \le i < j \le n-1$,

$$w(v_i v_j) = \alpha_i + \alpha_j, \tag{5.11}$$

where $w(v_i v_j) = 0$ if $v_i v_j \notin E(G)$. By (5.3), $|N(u)| \ge 3$. Let v_i, v_j, $v_k \in N(u)$. We have

$$w(uv_i) = \frac{1}{2}\left[(w(uv_i) + w(uv_j)) + (w(uv_i) + w(uv_k)) - (w(uv_j) + w(uv_k))\right].$$

By (5.8),

$$w(uv_i) = \frac{1}{2}\left[w(v_i v_j) + w(v_i v_k) - w(v_j v_k)\right].$$

If follows from (5.11) that

$$w(uv_i) = \frac{1}{2}[(\alpha_i + \alpha_j) + (\alpha_i + \alpha_k) - (\alpha_j + \alpha_k)] = \alpha_i.$$

Therefore, defining

$$T_i = T_i' + uv_i, \quad 1 \le i \le m,$$

we have

$$G = \sum_{i=1}^{m} \alpha_i T_i \,,$$

where T_i is a star tritree of G centered at v_i, $1 \le i \le m$. This completes the proof of Theorem 1.7. ∎

6. Path-Extremal Graphs

Definition 6.1. A weighted graph G on n vertices is *path-extremal* if its optimal paths are of weight precisely $2w(G)/n$.

Theorem 6.2. *A weighted graph is path-extremal if and only if it is a complete graph in which all edges have the same weight.*

Proof. Add a new vertex v_0 and join it to each vertex of G by an edge of weight M, where $M > w(G)$. The resulting graph G' is 2-connected and has weight

$$w(G') = w(G) + nM.$$

Let C' be an optimal cycle in G'. Then , by Theorem 1.3,

$$w(C') \ge \frac{2w(G')}{n} = \frac{2w(G)}{n} + 2M.$$

Since $M > w(G)$, vertex v_0 lies on C'. Let $P = C' - v_0$. Then

$$w(P) = w(C') - 2M \ge \frac{2w(G)}{n}.$$

But equality holds here because G is path-extremal. Consequently,

$$w(C') = \frac{2w(G')}{n}$$

and G' is cycle-extremal. By Theorem 1.7,

$$G' = \sum_{i=0}^{m} \alpha_i T_i'. \tag{6.1}$$

If G' has a 2-vertex cut $\{x, y\}$, then either $x = v_0$ or $y = v_0$. Thus, by Proposition 4.5,

$$M = w(xy) = \frac{w(G')}{n} = \frac{w(G)}{n} + M,$$

and $w(G) = 0$, a contradiction. If G' has no 2-vertex cut, then, by Proposition 3.2, T_i' is a spanning star of G', $0 \le i \le m$. Since $M > w(G)$, there must be some T_i' centered at v_0. We may suppose that T_0' is the star tritree centered at v_0. If $\alpha_0 = M$, then $m = 0$ and $w(G) = 0$, again a contradiction. Otherwise, $m = n$ and $\alpha_i = M - \alpha_0 > 0$, $1 \le i \le n$. Let $\alpha = \alpha_i$ and $T_i = T_i' - v_0v_i$, $1 \le i \le n$. It follows from (6.1) that $G = \sum_{i=1}^{n} \alpha T_i$, and so G is a uniformly weighted complete graph. ∎

7. Related Problems

In this section, an unweighted graph is regarded as a weighted graph G in which $w(e) = 1$ for every $e \in E(G)$. Thus, Definition 1.5 is still meaningful for unweighted graphs.

Conjecture 7.1. *Every 2-edge-connected graph has a collection* $\mathbf{C}$ *of cycles such that* $G = \sum_{C \in \mathbf{C}} \alpha_C C$ *and* $\sum_{C \in \mathbf{C}} \alpha_C \le (n-1)/2$

Remark 7.2. Conjecture 7.1 implies Theorem 1.3. To see this, let C^* be an optimal cycle of G. If Conjecture 7.1 is true, then

$$w(G) = \sum_{C \in \mathbf{C}} \alpha_C w(C) \le \left(\sum_{C \in \mathbf{C}} \alpha_C \right) w(C^*) \le \frac{n-1}{2} w(C^*),$$

which yields $w(C^*) \le 2w(G)/(n-1)$, as desired.

A *cycle double cover* of a graph G is a collection $\mathbf{C}$ of cycles of G such that each edge of G appears exactly twice in $\mathbf{C}$. Szekeres [8] and Seymour [6] have conjectured that every 2-edge-connected graph has a cycle double cover. A stronger conjecture is the following one.

Conjecture 7.3 [1]. *Every 2-edge-connected graph on* n *vertices has a cycle double cover* $\mathbf{C}$ *with* $|\mathbf{C}| \le n - 1$.

Remark 7.4. Conjecture 7.3 is stronger than Conjecture 7.1. Suppose that G has a cycle double cover $\mathbf{C}$ with $|\mathbf{C}| \le n-1$. Then

$$G = \sum_{C \in \mathbf{C}} \frac{1}{2} C \quad \text{and} \quad \sum_{C \in \mathbf{C}} \frac{1}{2} = \frac{|\mathbf{C}|}{2} \le \frac{n-1}{2}.$$

Remark 7.5. Since submission of this paper, Seymour [7] has proved Conjecture 7.1. His proof makes use of Theorem 1.3.

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